§5.2 Expected Values, Covariance, Correlation

Expected Values for joint p.m.f. are defined as usual

- $E[X] = \sum_{x,y} x \cdot p(x,y)$

  This is a double sum $= \sum_x \sum_y x \cdot p(x,y) = \sum_x x \sum_y p(x,y)$

  "μ_X" $= \sum_x x \cdot p_X(x)$

- $E[Y] = \sum_{x,y} y \cdot p(x,y)$

  "μ_Y" $= \sum_y y \cdot p_Y(y)$

- More generally, if $h(x,y)$ is any function, then

  $$E[h(X,Y)] = \sum_{x,y} h(x,y) \cdot p(x,y)$$

Examples:

- $E[X \cdot Y] = \sum_{x,y} x \cdot y \cdot p(x,y)$
- $E[X+Y] = \sum_{x,y} (x+y) \cdot p(x,y)$
- $E[\max(X,Y)] = \sum_{x,y} \max(x,y) \cdot p(x,y)$

If X & Y are continuous then sums ($\sum$) are replaced by integrals ($\iint \cdots dA$)

We can compute Variance with joint r.v. also

- $Var[X] = E[(X-\mu_X)^2]$ "spread in x direction"

  "σ_X^2" $= E[X^2] - (E[X])^2$

  These can be computed using the marginal pmf f_X also...

- $Var[Y] = E[(Y-\mu_Y)^2]$ "spread in y direction"

  "σ_Y^2" $= E[Y^2] - (E[Y])^2$

  These can be computed using the marginal pmf f_Y also...

> **Note:**
>
> - $E[X+Y] = E[X] + E[Y]$
>
>   More generally
>
> $E[aX + bY] = a\,E[X] + b\,E[Y]$

What about $Var[X+Y]$?

$$Var[X+Y] = E[(X+Y)^2 - (E[X+Y])^2]$$
$$= E[(X+Y)^2] - (E[X+Y])^2$$

$$\mathrm{Var}[X+Y] = E[(X+Y)^2] - (E[X+Y])^2$$
$$= E[X^2+2XY+Y^2] - (E[X]-E[Y])^2$$
$$= E[X^2] - (E[X])^2 \quad \mathrm{Var}[X]$$
$$+ 2E[XY] - 2E[X]\cdot E[Y]$$
$$+ E[Y^2] - (E[Y])^2 \quad \mathrm{Var}[Y]$$

Name the extra term above "Covariance":

Def: Covariance of X & Y is

$$\mathrm{Cov}[X,Y] = E[X\cdot Y - \mu_X \cdot \mu_Y]$$
"σ_{XY}"
$$= E[X\cdot Y] - E[X]\cdot E[Y]$$

Note: If X & Y are independent, then

$$p(x,y) = p_X(x)\cdot p_Y(y)$$

$$E[X\cdot Y] = \sum_x{}' \sum_y{}' x\cdot y\cdot p(x,y)$$
$$= \sum_x{}' \sum_y{}' x\cdot y\cdot p_X(x)\cdot p_Y(y)$$
$$= \left(\sum_x{}' x\cdot p_X(x)\right)\left(\sum_y{}' y\cdot p_Y(y)\right)$$
$$= E[X]\cdot E[Y]$$

Thm: If X & Y are independent then

- $\mathrm{Cov}[X,Y] = 0$
- $\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$

This probably will not happen if X & Y not indep.

Thm: If X & Y are not independent then

- $\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\cdot\mathrm{Cov}[X,Y]$

More generally

$$\mathrm{Var}[aX+bY] = a^2\,\mathrm{Var}[X] + b^2\,\mathrm{Var}[Y] + 2ab\,\mathrm{Cov}[X,Y]$$

(**Note:** $\mathrm{Cov}[X,a] = 0$

$\mathrm{Cov}[X,X] = \mathrm{Var}[X]$

$\mathrm{Cov}[aX,bY] = ab\cdot\mathrm{Cov}[X,Y]$)

Idea: $\mathrm{Cov}[X,Y]$ is "spread in diagonal $(x+y)$ direction"

Variance gives Standard Deviation

$$\sigma_X^2 = \mathrm{Var}[X] \rightsquigarrow \sigma_X = \sqrt{\mathrm{Var}[X]}$$

"Standard Deviation"

→ Standard Deviation scales with X:

$$\sqrt{\mathrm{Var}[aX]} = \sqrt{a^2\,\mathrm{Var}[X]}$$

$$\sigma_{aX} = |a| \cdot \sigma_X$$

"(Standard Deviation of aX) = $a \cdot$ (Standard Dev. of X)"

Covariance gives Correlation

$$\mathrm{Cov}[X,Y] \rightsquigarrow \rho = \frac{\mathrm{Cov}[X,Y]}{\sigma_X \cdot \sigma_Y}$$

"Correlation": Greek letter "rho"

- "rho" for "relation"
- letter before σ in Greek alphabet
- looks like sideways σ "sigma"

(Sometimes we will write "ρ_{XY}" to make it look more like σ_X & σ_Y)

→ Correlation is "unit free":

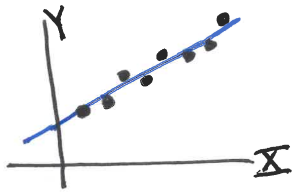
$$\frac{\mathrm{Cov}[aX, bY]}{\sigma_{aX} \cdot \sigma_{bY}} = \frac{ab\,\mathrm{Cov}[X,Y]}{(a\sigma_X)\cdot(b\,\sigma_Y)} = \frac{\mathrm{Cov}[X,Y]}{\sigma_X \cdot \sigma_Y}$$

Correlation Facts:

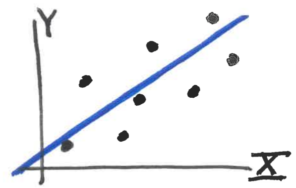
- $-1 \le \rho \le 1$
- X & Y independent $\Longrightarrow \rho = 0$
- $Y = aX$ (completely dependent) $\Longrightarrow \rho = \pm 1$ (same sign as a)

Idea: ρ measures how close the relationship between X & Y is to linear.

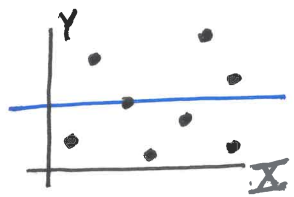
$\lvert\rho\rvert \ge .8$	"Strong Correlation"
$.5 < \lvert\rho\rvert < .8$	"Moderate Correlation"
$\lvert\rho\rvert \le .5$	"Weak Correlation"



Strong Correlation



Moderate Correlation



No Correlation

Example. Suppose X & Y have joint p.m.f.

$p(x,y)$	$X=-1$	0	1	$p_Y(y)$
$Y=2$	$2/15$	$2/15$	$1/15$	$5/15$
1	$2/15$	$3/15$	$2/15$	$7/15$
0	$1/15$	$1/15$	$1/15$	$3/15$
$p_X(x)$	$5/15$	$6/15$	$4/15$	

I) $E[X] = \sum_{x,y} x \cdot p(x,y)$

$= (-1)\left[2/15 + 2/15 + 1/15\right]$ ⟵ 1st column $x=-1$

$+ (0)\left[2/15 + 3/15 + 1/15\right]$ ⟵ 2nd column $x=0$

$+ (1)\left[1/15 + 2/15 + 1/15\right]$ ⟵ 3rd column $x=1$

$= (-1)\,5/15 + (0)\,6/15 + (1)\,4/15$

$= -1/15$

$E[Y] = \sum_y y \cdot p_Y(y)$ ⟵ Let's just use the marginal pmf this time

$= 0 \cdot 3/15 + 1 \cdot 7/15 + 2 \cdot 5/15$

$= 17/15$

$E[X^2] = \sum_x x^2 \cdot p_X(x)$

$= (-1)^2 \cdot 5/15 + 0^2 \cdot 6/15 + 1^2\, 4/15$

$= 9/15$

$E[Y^2] = \sum_y y^2\, p_Y(y)$

$= 0^2 \cdot 3/15 + 1^2 \cdot 7/15 + 2^2 \cdot 5/15$

$= 27/15$

$E[X \cdot Y] = \sum_{x,y} x \cdot y\; p(x,y)$ ⟵ cannot use marginals here...

$= (-1)(2) \cdot 2/15 + \cancel{(0)(2) \cdot 2/15} + (1)(2)\, 1/15$ ⟵ $=0$

$+ (-1)(1) \cdot 2/15 + \cancel{(0)(1) \cdot 3/15} + (1)(1)\, 2/15$

$\cancel{+ (-1)(0) \cdot 1/15 + (0)(0) \cdot 1/15 + (1)(0)\, 1/15}$ ⟵ $=0$

$= -4/15 + 2/15 - 2/15 + 2/15 = -2/15$

$E[X+Y] = E[X] + E[Y]$

$= -1/15 + 17/15 = 16/15$

Continuing Example

cannot use marginals here (or any other short-cut)

$$E[\min(X,Y)] = \sum_{x,y} \min(x,y)\cdot p(x,y)$$

$$\begin{aligned} &= (-1)\cdot \tfrac{2}{15} + \cancel{0\cdot \tfrac{2}{15}} + 1\cdot \tfrac{1}{15} \\ &\quad + (-1)\cdot \tfrac{2}{15} + \cancel{0\cdot \tfrac{3}{15}} + 1\cdot \tfrac{2}{15} \\ &\quad + (-1)\cdot \tfrac{1}{15} + \cancel{0\cdot \tfrac{1}{15}} + \cancel{0\cdot \tfrac{1}{15}} \\ &= -\tfrac{2}{15} \end{aligned}$$

The "dot product" of the "matrices" $\min(x,y)$ and $p(x,y)$

$$\begin{array}{cc|ccc} \min(x,y) & & & X & \\ & & -1 & 0 & 1 \\ \hline & 2 & -1 & 0 & 1 \\ Y & 1 & -1 & 0 & 1 \\ & 0 & -1 & 0 & 0 \end{array}$$

$\min(x,y)$ = smaller # of x & y

$$\begin{array}{cc|ccc} p(x,y) & & & X & \\ & & -1 & 0 & 1 \\ \hline & 2 & 2/15 & 2/15 & 1/15 \\ Y & 1 & 2/15 & 3/15 & 2/15 \\ & 0 & 1/15 & 1/15 & 1/15 \end{array}$$

probabilities.

II) Compute Variance and Covariance

$$Var[X] = E[X^2] - (E[X])^2 = \tfrac{9}{15} - \left(-\tfrac{1}{15}\right)^2 = \frac{134}{15^2} = \frac{134}{225}$$

But 15^2 is better in denom...

$$Var[Y] = E[Y^2] - (E[Y])^2 = \tfrac{27}{15} - \left(\tfrac{17}{15}\right)^2 = \frac{27(15) - 17^2}{15^2} = \frac{116}{15^2} = \frac{116}{225}$$

But 15^2 is better...

$$Cov[X,Y] = E[X\cdot Y] - E[X]\cdot E[Y] = -\tfrac{2}{15} - \left(-\tfrac{1}{15}\right)\left(\tfrac{17}{15}\right) = \frac{-13}{15^2}$$

III) Compute Standard Deviation & Correlation

$$\sigma_X = \sqrt{Var[X]} = \sqrt{\frac{134}{15^2}} = \frac{\sqrt{134}}{15}$$

$$\sigma_Y = \sqrt{Var[Y]} = \sqrt{\frac{116}{15^2}} = \frac{\sqrt{116}}{15}$$

$$\rho = \frac{Cov[X,Y]}{\sigma_X\cdot\sigma_Y} = \frac{-13/15^2}{\sqrt{134}/15\cdot\sqrt{116}/15} = -\frac{13}{\sqrt{134}\,\sqrt{116}}$$

IV) $Var[3X+Y] = 9\,Var[X] + Var[Y] + 6\,Cov[X,Y]$

$$= 9\cdot\frac{134}{15^2} + \frac{116}{15^2} + 6\left(\frac{-13}{15^2}\right)$$